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A MAXIMAL DEFINABLE σ-IDEAL OVER ω_1

BY

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ABSTRACT

The existence of a maximal definable nontrivial σ-ideal over ω_1 is equiconsistent with the existence of a measurable cardinal.

It is well known that there is no maximal nontrivial σ-ideal over ω_1, and the least cardinal that carries such an ideal is a measurable cardinal. On the other hand, the mathematical practice suggests that in the absence of additional hypotheses (such as $V = L$) one might not be able to define an extension of the ideal of nonstationary subsets of ω_1. (We recall that the axiom of determinacy implies that the ideal of thin sets is maximal. But then the axiom of choice fails of course.)

Thus we consider the statement

(1) "*there is a maximal definable nontrivial σ-ideal over ω_1*".

Below we show that if (1) holds then there is an inner model in which there is a measurable cardinal, and also, if there is a measurable cardinal, then there is a generic extension in which (1) holds. Hence the statement (1) is equiconsistent with the existence of a measurable cardinal.

The maximal definable ideal we obtain in the generic extension is not the thin ideal. Thus the question remains:

How strong is the consistency of

(2) "*the ideal of thin sets is a maximal definable ideal*"?

A few words about my indiscriminate use of the term "definable": this term is of course not expressible in the language of set theory and so, strictly speaking, the two theorems below are not theorems of set theory. However, it is possible to reformulate the statements (at the expense of clarity) so as to make them into

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formally correct sentences of ZFC. And the reader who is still uncomfortable can replace "definable" by "ordinal-definable" which is a perfectly legitimate property of sets, expressible in the language of ZF.

Throughout the paper we use the standard set theoretic terminology. We expect the reader to be familiar with the basic facts on measurable cardinals, σ-ideals, and forcing, in particular the Lévy collapse.

THEOREM 1. *Let I be a definable nontrivial σ-ideal over ω_1 and assume that there is no definable σ-ideal $J \supset I$. Then $L[U]$ exists.*

("$L[U]$ exists" means: there exist κ and $U \subset P(\kappa)$ such that in $L[U]$, κ is a measurable cardinal and U is a measure on κ.)

PROOF. We use one of the results from [1] on the core model K:

(3) *If there is a nontrivial elementary embedding of K then $L[U]$ exists.*

Let I be a definable nontrivial σ-ideal over ω_1 and let F be the dual filter. Let us assume that there is no definable σ-ideal $J \supset I$. I claim that

(4) if $X \subseteq \omega_1$ and $X \in K$ then either $X \in F$ or $\omega_1 - X \in F$.

To see that (4) holds, let X be the least (in the definable well ordering of K) subset of ω_1 such that (4) fails. Then the σ-ideal J generated by $I \cup \{X\}$ is definable, and $J \supset I$, contrary to the assumption.

Using F and functions on ω_1 that belong to K we form an ultrapower of K by F. Since I is σ-additive, the ultrapower is well founded and so we have an elementary embedding of K. But I is nontrivial and so the elementary embedding is nontrivial (as ω_1 is moved by the embedding). Now it follows from (3) that $L[U]$ exists. □

This proves one half of the equiconsistency result. We shall now prove the other half.

THEOREM 2. *Let us assume $V = L[D]$ where D is a normal measure on a measurable cardinal κ. There is a generic extension in which $\kappa = \aleph_1$ and the filter generated by D is a maximal definable σ-filter.*

PROOF. The generic extension is obtained by Lévy-collapsing κ to $\aleph_1$. If α is an inaccessible cardinal, we denote by B_α the *Lévy algebra*, the complete Boolean algebra corresponding to the notion of forcing introduced by Lévy in [3], which collapses all uncountable cardinals below α and makes $\alpha = \aleph_1$. B_α satisfies the α-chain condition, has size α, and if α is an inaccessible limit of inaccessible cardinals, then B_α is the direct limit of the B_β, $\beta < \alpha$. The important property of Lévy algebras that we are going to use is the *homogeneity* of B_α, which is implicit in Solovay's work [4]:

(4) *if A is a complete Boolean subalgebra of B_α and $|A| < \alpha$, then every automorphism of A can be extended to an automorphism of B_α.*

Thus let $\mathcal{M}$ denote the ground model and assume that $\mathcal{M}$ satisfies $V = L[D]$ where D is a normal measure on κ. Let B_κ be the Lévy algebra for making $\kappa = \aleph_1$, and let G be a generic ultrafilter on B_κ.

In $\mathcal{M}[G]$, let $\bar{D}$ be the filter over $\aleph_1$ generated by D. Since D is κ-complete (in $\mathcal{M}$) and B_κ has the κ-chain condition, $\bar{D}$ is a σ-complete filter. It is nontrivial as D is nonprincipal, and it is definable since D is the unique normal measure in the unique model $L[D]$ (cf. [2]). We shall show that there is (in $\mathcal{M}[G]$) no definable filter F such that $F \supset \bar{D}$. Thus $\bar{D}$ is a maximal definable σ-filter in $M[G]$.

Let F be a B_κ-valued name and assume that for some condition $b \in B_\kappa$,

$$b \Vdash F \text{ is a filter extending } \bar{D}, \tag{5}$$

that for some X

$$b \Vdash X \in F \quad \text{and} \quad X \notin \bar{D} \tag{6}$$

and that φ is a definition of F, i.e.

$$b \Vdash F \text{ is the unique } F \text{ such that } \varphi(F). \tag{7}$$

In order to reach a contradiction, it suffices if we find an automorphism π of B_κ such that

$$\pi(b) = b, \tag{8}$$

and

$$b \Vdash F \neq \pi(F). \tag{9}$$

Since (7), (8) and (9) cannot hold simultaneously, we have a contradiction.

Since B_κ is the direct limit of the B_α, $\alpha < \kappa$, we may assume that

$$B_\kappa = \bigcup_\alpha B_\alpha, \tag{10}$$

the union in (10) being taken over all inaccessible $\alpha < \kappa$, and $B_\alpha \subset B_\beta$ whenever $\alpha < \beta$ (and $B_\alpha = \bigcup_\beta B_\beta$ when α is an inaccessible limit of inaccessibles). Let S_0 be the set of all inaccessible $\alpha < \kappa$ such that $b \in B_\alpha$. In order to simplify the arguments, we shall assume that $b = 1$. The proof in the general case is similar, only slightly more complicated, or it can be derived from the special case directly, as follows: the Boolean algebra $B_\kappa \restriction b = \{a : a \leqq b\}$ is isomorphic to B_κ and so the construction of π below can be carried out for $B_\kappa \restriction b$, and then π can be extended trivially to B_κ.

For each $\alpha \in S_0$, let

$$b_\alpha = \| \alpha \in X \|, \tag{11}$$

and let

$$\begin{aligned} a_\alpha &= \Sigma\{a \in B_\alpha : a \leqq b_\alpha\}, \\ c_\alpha &= \Sigma\{c \in B_\alpha : c \leqq -b_\alpha\}. \end{aligned} \tag{12}$$

For each $\alpha \in S_0$ we have $a_\alpha, c_\alpha \in B_\alpha$. Now we use the fact that D is a normal measure (and $S_0 \in D$), and the property (10) together with the fact that $|B_\alpha| = \alpha$ for all $\alpha \in S_0$. It follows that there exist $a, c \in B_\kappa$ and a set $S \in D$ such that

$$a_\alpha = a, \qquad c_\alpha = c \tag{13}$$

for all $\alpha \in S$.

Now if $a \neq 0$ then $a \Vdash S \subseteq X$ and so

$$a \Vdash X \in \bar{D} \tag{14}$$

contrary to (6). If $c \neq 0$ then, similarly,

$$c \Vdash \kappa - X \in \bar{D} \tag{15}$$

but that contradicts (5). Hence $a = c = 0$.

Using (10), we can find a closed unbounded set $C \subseteq \kappa$ such that

(16) (i) if α is a successor point of C then $\alpha \in S$,
(ii) if $\alpha \in C$, $\beta < \alpha$ and $\beta \in S$ then $b_\beta \in B_\alpha$.

We shall now construct an automorphism π of B_κ which satisfies (9). We construct π as the union of automorphisms π_α of B_α, for $\alpha \in C$, where $\pi_\alpha \subset \pi_\beta$ whenever $\alpha < \beta$.

We construct the π_α's by induction on α. If α is a limit point of C then we let $\pi_\alpha = \bigcup_{\beta<\alpha} \pi_\beta$ (if α is not inaccessible, then we let B_α be the completion of $\bigcup_{\beta<\alpha} B_\beta$, and π_α the unique extension of $\bigcup_{\beta<\alpha} \pi_\beta$ to B_α).

Let $\alpha \in C$ be a successor point of C, and let β be the largest $\beta \in C$ below α. We have already constructed π_β. If $\beta \notin S$ then we extend π_β arbitrarily to an automorphism π_α of B_α, using (4); this is possible because α is inaccessible. Thus assume that $\beta \in S$. Let A be the complete subalgebra of B_α generated by $B_\beta \cup \{b_\beta\}$. Every element of A has the form

$$x \cdot b_\beta + y \cdot - b_\beta \tag{17}$$

for some $x, y \in B_\beta$. Since $\beta \in S$, we have $a_\beta = c_\beta = 0$ and so there are no nonzero elements $u, v \in B_\beta$ such that $u \leqq b_\beta$, $v \leqq - b_\beta$. It follows that the representation (17) is unique. Thus if we define

$$\pi(x \cdot b_\beta + y \cdot - b_\beta) = \pi_\beta(x) \cdot - b_\beta + \pi_\beta(y) \cdot b_\beta \tag{18}$$

the uniqueness of (17) guarantees that π is well-defined for all elements of A, and that π is an automorphism of A extending π_β.

Now we use (4) to extend π to an automorphism π_α of B_α. Note that $\pi_\alpha \supset \pi_\beta$, and

$$\pi_\alpha(b_\beta) = - b_\beta. \tag{19}$$

Finally, we let $\pi = \bigcup_{\alpha \in C} \pi_\alpha$; π is an automorphism of B_κ, and

$$\pi(b_\beta) = - b_\beta \tag{20}$$

for all $\beta \in C \cap S$. It follows that for all $\beta \in C \cap S$,

$$1 \Vdash \beta \in X \leftrightarrow \beta \notin \pi(X). \tag{21}$$

Since $S \cap C \in D$, it follows from (5) that

$$1 \Vdash X \cap S \cap C \in F.$$

This, together with (21) and because

$$1 \Vdash \pi(X) \in \pi(F),$$

implies that

$$1 \Vdash F \neq \pi(F).$$

This completes the proof of (9) in the special case when $b = 1$. The general case, proved similarly, provides the desired contradiction showing that $\bar{D}$ has no definable proper extension in $M[G]$. □

We conclude by reformulating the problem mentioned at the beginning of the article.

PROBLEM. Is the following statement consistent with ZFC (relative to "there exists a measurable cardinal")?

"*The closed unbounded filter is the only definable normal filter over* $\aleph_1$".

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